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Properties of the Log-Barrier Function on Degenerate Nonlinear Programs

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Abstract. We examine the sequence of local minimizers of the log-barrier function for a nonlinear program near a solution at which second-order sufficient conditions and the Mangasarian-Fromovitz constraint qualifications are satisfied, but the active constraint gradients are not necessarily linearly independent. When a strict complementarity condition is satisfied, we show uniqueness of the local minimizer of the barrier function in the vicinity of the nonlinear program solution, and obtain a semi-explicit characterization of this point. When strict complementarity does not hold, we obtain several other interesting characterizations, in particular, an estimate of the distance between the minimizers of the barrier function and the nonlinear program in terms of the barrier parameter, and a result about the direction of approach of the sequence of minimizers of the barrier function to the nonlinear programming solution.

1. Introduction

We consider the nonlinear programming problem

$$\min f(x) \qquad \text{subject to} \qquad c(x) \geq 0, \tag{1}$$

where $f : \mathsf{R}^n \to \mathsf{R}$ and $c : \mathsf{R}^n \to \mathsf{R}^m$ are smooth (twice Lipschitz continuously differentiable) functions. We assume that second-order sufficient conditions hold at a point x^*, so that x^* is a strict local solution of (1).

The logarithmic barrier function for (1) is

$$P(x;\mu) = f(x) - \mu \sum_{i=1}^{m} \log c_i(x). \tag{2}$$

Under conditions assumed in this paper, and discussed in detail below, this function has a local minimizer near x^* for all μ sufficiently small. Methods based on (2) find approximations to the minimizer of $P(\cdot;\mu_k)$, which we denote by

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$x(\mu_k)$, for some sequence $\{\mu_k\}$ with $\mu_k \downarrow 0$, usually by applying some variant of Newton's method. Extrapolation techniques are sometimes used to find an appropriate initial guess after each change in μ.

In this paper, we examine properties of the sequence of minimizers of $P(\cdot;\mu)$ for small μ, in the vicinity of x^*. Previous analyses have assumed that the active constraint gradients are linearly independent at the solution—the so-called linear independence constraint qualification. By constrast, we make the weaker assumption that the Mangasarian-Fromovitz constraint qualification holds. This more general condition, which is equivalent to boundedness of the set of optimal Lagrange multipliers, has been used by many authors in studying the local convergence analysis of nonlinear optimization and complementarity problems and the stability of their solutions. In Section 3, we examine the case in which at least one of the optimal multipliers satisfies the strict complementarity condition (defined in the next section). In this case, the path of minimizers of $P(\cdot;\mu)$ behaves similarly to the case of linearly independent constraints: The minimizers are locally unique, the path traced by the minimizers is smooth (as a function of μ) with a well-defined derivative, and the corresponding sequence of multiplier estimates approaches the analytic center of the multiplier set. In Section 4, we consider the case in which the strict complementarity condition does not hold. In this case, the path traced by the set of minimizers takes on a completely different character. We prove an existence result, derive an estimate of the distance between the minimizer of $P(\cdot;\mu_k)$ in terms of μ_k, and show that any path of minimizers that converges to x^* must approach this point tangentially to the strongly active constraints.

The previous literature on the log-barrier function and properties of the minimizers of $P(\cdot;\mu)$ is plentiful. The seminal book of Fiacco and McCormick [8] presents general results about the existence of minimizers of the barrier function in the vicinity of x^* and the convergence of the minimizer sequence to x^* as $\mu_k \downarrow 0$ [8, Theorem 8]. It also shows that the path of minimizers of $P(\cdot;\mu)$ is isolated and smooth when the active constraint gradients are linearly independent and strict complementarity holds [8, Sections 5.1, 5.2]. Adler and Monteiro [1] analyze the trajectories produced by minimizers of the log-barrier function in the case of linear programming. The differences in formulation and the linearity of the problem make it difficult to relate the results of Adler and Monteiro to those of this paper. However, their Theorem 3.2 corresponds to our observation that the Lagrange multiplier estimates converge to the analytic center of the optimal multiplier set, while their Theorem 5.4 corresponds to our Theorem 3 in describing the direction of approach of the trajectory of minimizers to x^*. The lack of curvature in their problem gives the results a significantly different flavor, however, and their proof techniques depend strongly on the constancy of the closed subspace spanned by the active constraint gradients in the vicinity of x^*, which is not the case in (1) under our assumptions.

Although their focus is on the log-barrier function, Fiacco and McCormick [8] actually consider a more general class of barrier functions, and also derive results for the case in which equality constraints are represented by quadratic penalty terms. Nesterov and Nemirovskii [17] study the general class of self-concordant barriers of which the log barrier is a particular instance. Following the results of Murray [16] and Lootsma [14] regarding the ill conditioning of the Hessian matrix $P_{xx}(\cdot;\mu)$ along the central path, the nature of the ill conditioning in the neighborhood of the solution is examined further by M. H. Wright [23]. The latter paper proposes techniques for calculating approximate Newton steps for the function $P(\cdot;\mu)$ that do not require the solution of ill-conditioned systems. In earlier work, Gould [12] proposed a method for computing accurate Newton steps by identifying the active indices explicitly, and forming an augmented linear system that remains well conditioned even when μ is small. The effect of finite-precision arithmetic on the calculation of Newton steps is examined by M. H. Wright [24]. Both M. H. Wright [23,24] and S. J. Wright [26] use a subspace decomposition of the Hessian $P_{xx}(\cdot;\mu)$ like the one used in Section 3 below, but there is an important distinction that we note later. The paper [26] addresses the issue of domain of convergence of Newton's method applied to $P(\cdot;\mu)$, which is also addressed in Theorem 1 below.

The Mangasarian-Fromovitz constraint qualification has been used in place of the standard assumption of linear independence of the constraint gradients in several recent works on nonlinear programming. Ralph and Wright [20] describe a path-following method for convex nonlinear programming that achieves superlinear local convergence under this condition. S. J. Wright [28,25] and Anitescu [2] study the local convergence of sequential quadratic programming methods under this assumption.

Our paper concludes with comments about two important issues: Convergence of the Newton/log-barrier method, in which Newton's method is used to find an approximate minmizer of $P(\cdot;\mu_k)$ for each μ_k, and relevance of our results to primal-dual methods, which generate iterates with both primal and dual (Lagrange multiplier) components rather than primal components alone. Detailed study of these topics is left to future work.

2. Assumptions, Notation, and Basic Results

2.1. Assumptions

In this section, we specify the optimality conditions for the nonlinear program (1) and outline our assumptions on the solution x^*.

Assume first that the functions f and c are twice Lipschitz continuously differentiable in the neighborhood of interest. The Lagrangian function for (1) is

$$\mathcal{L}(x,\lambda) = f(x) - \lambda^T c(x), \tag{3}$$

where λ is the vector of Lagrange multipliers. Necessary conditions for x^* to be a solution of (1) are that there exists a Lagrange multiplier vector λ^* such that

$$c(x^*) \geq 0, \quad \lambda^* \geq 0, \quad (\lambda^*)^T c(x^*) = 0, \quad \mathcal{L}_x(x^*,\lambda^*) = 0. \tag{4}$$

The active constraints are the components of c for which $c_i(x^*) = 0$. Without loss of generality we assume these to be the first q components of c, so that

$$c_i(x^*) = 0 \quad i = 1, 2, \ldots, q, \tag{5a}$$
$$c_i(x^*) > 0, \ \lambda_i^* = 0, \quad i = q+1, \ldots, m. \tag{5b}$$

We define U_R to be an orthonormal matrix of dimensions $n \times \bar{q}$ for some $\bar{q} \le q$ whose columns span the range space of the active constraint gradients, that is,

$$\text{Range } U_R = \text{Range } \{\nabla c_i(x^*) \,|\, i = 1, 2, \ldots, q\}. \tag{6}$$

We let U_N denote an orthonormal matrix of dimensions $n \times (n - \bar{q})$ whose columns span the space of vectors orthogonal to $\nabla c_i(x^*)$ for all $i = 1, 2, \ldots, q$. By the fundamental theorem of algebra, we have that

$$\left[\, U_R \; U_N \,\right] \text{ is orthogonal.} \tag{7}$$

We assume that the *Mangasarian-Fromovitz constraint qualification (MFCQ)* holds at x^*, which is that there is a vector p such that

$$\nabla c_i(x^*)^T p < 0, \quad i = 1, 2, \ldots, q. \tag{8}$$

The stronger linear independence constraint qualification (LICQ), which assumes linear independence of the vectors $\nabla c_i(x^*)$, $i = 1, 2, \ldots, q$, is used by M. H. Wright [22], Fiacco and McCormick [8], and S. J. Wright [26], for instance. Unlike LICQ, MFCQ does not imply uniqueness of λ^*. We can use (3) and (4) to express the conditions on λ^* as

$$\nabla f(x^*) = \textstyle\sum_{i=1}^q \lambda_i^* \nabla c_i(x^*), \quad \lambda_i^* \ge 0, \ i = 1, 2, \ldots, q, \tag{9a}$$
$$\lambda_i^* = 0, \ i = q+1, \ldots, m. \tag{9b}$$

We define $\mathcal{S}_\lambda$ to be the set of multipliers satisfying these conditions at x^*, that is,

$$\mathcal{S}_\lambda \stackrel{\triangle}{=} \{\lambda^* \,|\, (x^*, \lambda^*) \text{ satisfy (4)}\}. \tag{10}$$

Gauvin [10, Theorem 1] shows that the condition (8) is equivalent to boundedness of the set $\mathcal{S}_\lambda$. We conclude from (9) that $\mathcal{S}_\lambda$ is a bounded polyhedral set.

The *strict complementarity* condition is that

$$\lambda_i^* + c_i(x^*) > 0, \qquad i = 1, 2, \ldots, m, \tag{11}$$

for at least one $\lambda^* \in \mathcal{S}_\lambda$. This condition is assumed in Section 3. When it holds, we can define the analytic center $\bar{\lambda}^*$ of $\mathcal{S}_\lambda$ to be the vector that solves

$$\min_{\lambda^*} - \sum_{i=1}^q \ln \lambda_i^*, \tag{12}$$

over the set of strictly complementary multipliers, that is,

$$\nabla f(x^*) = \sum_{i=1}^{q} \lambda_i^* \nabla c_i(x^*), \quad \lambda_i^* > 0,\ i = 1, 2, \ldots, q, \tag{13a}$$
$$\lambda_i^* = 0,\ i = q+1, \ldots, m. \tag{13b}$$

Since the problem (12), (13) has a smooth, strictly convex objective and a bounded feasible set, it has a unique minimizer $\bar{\lambda}^*$ whose components $1, 2, \ldots, q$ are characterized by the first-order conditions, which is that there exists a vector $\zeta \in \mathrm{R}^n$ such that

$$\frac{1}{\bar{\lambda}_i^*} = \nabla c_i(x^*)^T \zeta > 0, \quad i = 1, 2, \ldots, q. \tag{14}$$

(Were MFCQ *not* satisfied, there would not exist a vector ζ such that $\nabla c_i(x^*)^T \zeta > 0$, $i = 1, 2, \ldots, q$, and so the problem (12), (13) would have no solution.) Note that ζ is defined by (14) only up to a term in the null space of the active constraint gradients. In other words, if we decompose ζ as

$$\zeta = U_R \zeta_R + U_N \zeta_N, \tag{15}$$

where U_R and U_N are defined as in (6), (7), the formula (14) defines ζ_R uniquely while leaving ζ_N completely free.

Finally, we assume that second-order sufficient conditions for optimality are satisfied, that is,

$$y^T \mathcal{L}_{xx}(x^*, \lambda^*) y > 0, \quad \text{for all } \lambda^* \in \mathcal{S}_\lambda \tag{16}$$
$$\text{and all } y \neq 0 \text{ with } \nabla c_i(x^*)^T y = 0 \text{ for all } i = 1, 2, \ldots, q.$$

Using the matrix U_N defined in (7), we note that this condition is equivalent to

$$U_N^T \mathcal{L}_{xx}(x^*, \lambda^*) U_N \text{ positive definite, for all } \lambda^* \in \mathcal{S}_\lambda. \tag{17}$$

Note that this condition, together with (4) and strict complementarity, implies that x^* is a strict local solution of the problem (1); see, for example, Nocedal and Wright [18, Theorem 12.6]. Using the condition (17) together with the decomposition (15) of ζ, we can define a particularly interesting value of ζ_N to be the (unique) solution of the following system:

$$U_N^T \mathcal{L}_{xx}(x^*, \bar{\lambda}^*) U_N \zeta_N = -U_N^T \mathcal{L}_{xx}(x^*, \bar{\lambda}^*) U_R \zeta_R + \sum_{i=q+1}^{m} \frac{1}{c_i(x^*)} U_N^T \nabla c_i(x^*), \tag{18}$$

where ζ_R is defined uniquely by the condition (14). The properties of the value of ζ_N defined by this formula become apparent in Section 3.

2.2. Notation

We use the following notation in the rest of the paper. For related positive quantities α and β, we say $\beta = O(\alpha)$ if there is a constant M such that $\beta \le M\alpha$ for all α sufficiently small. We say that $\beta = o(\alpha)$ if $\beta/\alpha \to 0$ as $\alpha \to 0$, $\beta = \Omega(\alpha)$ if $\alpha = O(\beta)$, and $\beta = \Theta(\alpha)$ if $\beta = O(\alpha)$ and $\alpha = O(\beta)$. It follows that the expression $\beta = O(1)$ means that $\beta \le M$ for some constant M and all values of β in the domain of interest.

We say that the function $\phi : [0,\infty) \to [0,\infty)$ is *positively homogeneous* if it is continuous, satisfies $\phi(0) = 0$ and $\phi(\tau) > 0$ for $\tau > 0$, and is increasing on $[0,\infty)$.

For a given value of μ, we define a local minimizer of $P(\cdot;\mu)$ close to x^* generically by $x(\mu)$. (The uniqueness or at least specialness of this point is made clear in subsequent discussions.)

2.3. Basic Results

Given any strictly feasible point x and any positive value of the barrier parameter μ in (2), we define a vector of Lagrange multiplier estimates $\lambda(x,\mu)$ by

$$\lambda(x,\mu) = \mu C(x)^{-1} e = \left[\frac{\mu}{c_1(x)}, \ldots, \frac{\mu}{c_m(x)} \right]^T. \tag{19}$$

The derivatives of the barrier function (2) are

$$P_x(x;\mu) = \nabla f(x) - \sum_{i=1}^m \frac{\mu}{c_i(x)} \nabla c_i(x), \tag{20a}$$

$$P_{xx}(x;\mu) = \nabla^2 f(x) + \mu \sum_{i=1}^m \left[\frac{1}{c_i^2(x)} \nabla c_i(x) \nabla c_i(x)^T - \frac{1}{c_i(x)} \nabla^2 c_i(x) \right]. \tag{20b}$$

By combining (19) with (20a), we obtain

$$\nabla f(x) = \sum_{i=1}^m \lambda_i(x,\mu) \nabla c_i(x) + P_x(x;\mu), \tag{21}$$

while for the case $x = x(\mu)$, we have from (19) that

$$\nabla f(x(\mu)) = \sum_{i=1}^m \lambda_i(x(\mu),\mu) \nabla c_i(x(\mu)).$$

We denote by $\mathcal{C}$ the feasible set for (1), that is,

$$\mathcal{C} \triangleq \{x \mid c(x) \ge 0\},$$

and by strict $\mathcal{C}$ we denote the set of points at which the inequalities are satisfied strictly, that is,

$$\text{strict } \mathcal{C} \stackrel{\Delta}{=} \{x \mid c(x) > 0\}.$$

It is easy to show that, under the MFCQ assumption (8), there is a neighborhood of x^* within which strict $\mathcal{C}$ coincides with int $\mathcal{C}$.

Lemma 1. *Suppose that (8) is satisfied. Then there is a neighborhood $\mathcal{N}$ of x^* such that*

$$\text{strict } \mathcal{C} \cap \mathcal{N} = \text{int } \mathcal{C} \cap \mathcal{N} \neq \emptyset,$$

and x^* lies in the closure of strict $\mathcal{C}$.

Proof. Choose $\mathcal{N}$ such that (8) continues to hold whenever x^* is replaced by x, for all $x \in \mathcal{C} \cap \mathcal{N}$, while the constraints $q+1, \ldots, m$ remain inactive throughout $\mathcal{N}$. We prove the result by showing that strict $\mathcal{C} \cap \mathcal{N} \subset$ int $\mathcal{C} \cap \mathcal{N}$, and then the converse.

Consider some $x \in$ strict $\mathcal{C} \cap \mathcal{N}$. By continuity of c, we can choose $\delta > 0$ such that the open Euclidean ball of radius δ around x, denoted by $\mathcal{B}_\delta(x)$, satisfies $\mathcal{B}_\delta(x) \subset \mathcal{N}$ and $c(z) > 0$ for all $z \in \mathcal{B}_\delta(x)$. Hence, $z \in$ strict $\mathcal{C} \cap \mathcal{N} \subset \mathcal{C}$, and therefore $x \in$ int $\mathcal{C}$.

Now consider a point $x \in \mathcal{N} \backslash$strict $\mathcal{C}$. If $x \notin \mathcal{C}$, then clearly $x \notin$ int $\mathcal{C} \cap \mathcal{N}$, and we are done. Otherwise, we have $c_j(x) = 0$ for some $j \in \{1, 2, \ldots, q\}$. Consider now the points $x - \alpha p$ for p defined in (8) and α small and positive. We have by continuity of ∇c_j that

$$c_j(x - \alpha p) = c_j(x) - \alpha \nabla c_j(x)^T p + o(\alpha) < 0,$$

for all $\alpha > 0$ sufficiently small. Therefore, $x - \alpha p \notin \mathcal{C}$, so that $x \notin$ int $\mathcal{C}$.

The claim that strict $\mathcal{C} \cap \mathcal{N} \neq \emptyset$ is proved by considering points of the form $x^* + \alpha p$, for $\alpha > 0$ and p satisfying (8). Consideration of the same set of points demonstrates that x^* lies in the closure of strict $\mathcal{C}$. ∎

We now show boundedness of the Lagrange multiplier estimates arising from approximate minimization of $P(\cdot; \mu)$.

Lemma 2. *Suppose that the first-order necessary conditions (4) and the MFCQ condition (8) hold. Then, given any $\beta_1 \geq 0$, there exist positive quantities β_2 and ϵ such that for all barrier parameters $\mu \in (0, \epsilon]$ and all strictly feasible x that satisfy $\|P_x(x; \mu)\| \leq \beta_1$, we have that the multipliers $\lambda(x, \mu)$ defined by (19) satisfy $\|\lambda(x, \mu)\| \leq \beta_2$.*

Proof. Suppose for contradiction that there exist sequences $\{\mu_k\}$ and $\{x^k\}$ such that $\mu_k \downarrow 0$ and $\|P_x(x^k; \mu_k)\| \leq \beta_1$, but $\|\lambda(x^k, \mu_k)\| \uparrow \infty$. By dividing (20a) (with $x = x^k$ and $\mu = \mu_k$) by $\|\lambda(x^k, \mu_k)\|$ and defining

$$\bar{\lambda}^k = \frac{\lambda(x^k, \mu_k)}{\|\lambda(x^k, \mu_k)\|},$$

we have that

$$0 = \lim_{k\to\infty} \|\lambda(x^k,\mu_k)\|^{-1} P_x(x_k;\mu_k) \tag{22}$$

$$= \sum_{i=1}^{m} \bar{\lambda}_i^k \nabla c_i(x^k) + O(\|\lambda(x^k,\mu_k)\|^{-1}) \tag{23}$$

$$= \sum_{i=1}^{m} \bar{\lambda}_i^k \nabla c_i(x^*) + O(\|\lambda(x^k,\mu_k)\|^{-1}) + O(\|x^k - x^*\|). \tag{24}$$

Note that $\bar{\lambda}^k \geq 0$ for each k. By taking a subsequence if necessary, we have by compactness of the unit ball that there is a vector $\bar{\lambda} \geq 0$ with $\|\bar{\lambda}\| = 1$ such that $\bar{\lambda}^k \to \bar{\lambda}$. Hence, by taking the limit of the expression above, we have that

$$\sum_{i=1}^{m} \bar{\lambda}_i \nabla c_i(x^*) = 0.$$

By taking inner products with the vector p from (8), we have that

$$\sum_{i=1}^{m} \bar{\lambda}_i (\nabla c_i(x^*)^T p) = 0.$$

Since each coefficient $\nabla c_i(x^*)^T p$ is strictly positive, we have from $\bar{\lambda} \geq 0$ that $\bar{\lambda} = 0$, a contradiction. Hence no such sequences $\{\mu_k\}$ and $\{x_k\}$ exist, and the result is proved. ∎

A slight extension of this result shows conditions under which the sequence of multiplier estimates converges to the optimal multiplier set $\mathcal{S}_\lambda$.

Lemma 3. *Suppose that the first-order necessary conditions (4) and the MFCQ condition (8) hold. Then, given any sequences* $\{\mu_k\}$ *and* $\{x^k\}$ *with* $x^k \to x^*$, $\mu_k \downarrow 0$, *and* $P_x(x^k;\mu_k) \to 0$, *the sequence of multiplier estimates* $\lambda(x^k,\mu_k)$ *defined by (19) satisfies*

$$\operatorname{dist}_{\mathcal{S}_\lambda} \lambda(x^k,\mu_k) = O\left(\left\|x^k - x^*\right\|\right) + O(\mu_k) + O\left(\left\|P_x(x^k;\mu_k)\right\|\right).$$

Proof. From Lemma 2, we have that the sequence $\lambda(x^k;\mu_k)$ is bounded. Therefore, we have

$$\begin{aligned} P_x(x^k;\mu_k) &= \nabla f(x^k) - \sum_{i=1}^{m} \frac{\mu_k}{c_i(x_k)} \nabla c_i(x^k) \\ &= \nabla f(x^*) - \sum_{i=1}^{q} \lambda(x^k,\mu_k) \nabla c_i(x^*) + O(\mu_k) + O(\|x^k - x^*\|). \end{aligned}$$

By comparing this expression with the definition (9), (10) of $\mathcal{S}_\lambda$, and applying Hoffmann's lemma [13], we obtain the desired result. ∎

3. Behavior of the Central Path near x^* Under Strict Complementarity

In this section, we examine the properties of a path of exact minimizers $x(\mu)$ of $P(x;\mu)$. We will refer to the set $\{x(\mu), \mu \geq 0\}$ as the *primal central path.*

Our main result, Theorem 1, shows the existence of a minimizer $x(\mu)$ of $P(\cdot;\mu)$ that lies within a distance $\Theta(\mu)$ of x^*, and characterizes the domain of convergence of Newton's method for $P_x(\cdot;\mu)$ to this minimizer. It derives a first-order estimate of the location of this minimizer, showing that it lies within a distance $O(\mu^2)$ of the point

$$\bar{x}(\mu) = x^* + \mu\zeta, \tag{25}$$

where $\mu > 0$ and ζ is the vector that is uniquely specified by the formulae (14), (15), and (18). Our second result, Theorem 2, shows that the minimizer $x(\mu)$ is locally unique in a certain sense.

One key to the analysis is the partitioning of $\mathbb{R}^n$ into two orthogonal subspaces, defined by the matrices U_R and U_N of (6) and (7). This decomposition was also used in the analysis of S. J. Wright [27], but differs from those used by M. H. Wright in [23, 24] and S. J. Wright [26], which define these matrices with respect to the active constraint matrix evaluated at the current iterate x, rather than at the solution x^* of (1). By using the latter strategy, we avoid difficulties with loss of rank in the active constraint matrix at the solution, which may occur under the MFCQ assumption of this paper, but not under the LICQ assumption used in [23, 24, 26].

All results in this section use the following assumption.

Assumption 1. *The first-order necessary conditions (4), the second-order sufficient conditions (16), the MFCQ (8), and the strict complementarity condition (11) hold at x^*.*

Our first lemma concerns the length of a Newton step for $P(\cdot;\mu)$, taken from a point x that is close to $\bar{x}(\mu)$.

Lemma 4. *Suppose that Assumption 1 holds. Given some fixed $\sigma \in (1, 2]$, there is a radius $\rho > 0$ such that for any $\rho_0 \in (0, \rho]$ the following property holds: There is a quantity $C_2 > 0$ depending on ρ_0 such that for all x with*

$$\|x - (x^* + \mu\zeta)\| = \|x - \bar{x}(\mu)\| \leq \rho_0\mu^\sigma, \tag{26}$$

with $\mu \in (0, 1]$, the Newton step s generated from x satisfies

$$\|s\| \leq C_2\mu^\sigma.$$

The dependence of C_2 on ρ_0 is positively homogeneous.

Proof. In the following analysis, we frequently use the order notation $O(\cdot)$ and $\Theta(\cdot)$, keeping in mind that the constant multipliers that are hidden in these expressions can be made as small as we like by decreasing ρ_0.

We first examine the properties of $P_x(x;\mu)$ by expanding about x^* and using the properties (26) and (5) to obtain

$$\begin{aligned}
&P_x(x;\mu)\\
&= \nabla f(x^*) + \mu \nabla^2 f(x^*)\zeta + O(\mu^\sigma)\\
&\quad - \sum_{i=1}^{q} \mu \left[\mu \nabla c_i(x^*)^T \zeta + O(\mu^\sigma)\right]^{-1} \left[\nabla c_i(x^*) + \mu \nabla^2 c_i(x^*)\zeta + O(\mu^\sigma)\right]\\
&\quad - \sum_{i=q+1}^{m} \mu \left[c_i(x^*) + O(\mu)\right]^{-1} \left[\nabla c_i(x^*) + O(\mu)\right].
\end{aligned}$$

Using (14) and the estimate $[1+O(\mu^{\sigma-1})]^{-1} = 1 + O(\mu^{\sigma-1})$ (which holds for all ρ_0 sufficiently small), we obtain

$$\begin{aligned}
&P_x(x;\mu)\\
&= \nabla f(x^*) + \mu \nabla^2 f(x^*)\zeta + O(\mu^\sigma)\\
&\quad - \sum_{i=1}^{q} \bar\lambda_i^* \left[1 + \bar\lambda_i^* O(\mu^{\sigma-1})\right]^{-1} \left[\nabla c_i(x^*) + \mu \nabla^2 c_i(x^*)\zeta + O(\mu^\sigma)\right]\\
&\quad - \sum_{i=q+1}^{m} \frac{\mu}{c_i(x^*)} \nabla c_i(x^*) + O(\mu^2)\\
&= \nabla f(x^*) - \sum_{i=1}^{q} \bar\lambda_i^* \nabla c_i(x^*) + \mu \left[\nabla^2 f(x^*) - \sum_{i=1}^{q} \bar\lambda_i^* \nabla^2 c_i(x^*)\right]\zeta\\
&\quad + O(\mu^{\sigma-1}) \sum_{i=1}^{q} \left(\bar\lambda_i^*\right)^2 \nabla c_i(x^*) - \sum_{i=q+1}^{m} \frac{\mu}{c_i(x^*)} \nabla c_i(x^*) + O(\mu^\sigma).
\end{aligned}$$

Hence by the definition (3) and the first-order conditions (4), we have

$$\begin{aligned}
P_x(x;\mu) = \mu \mathcal{L}_{xx}(x^*, \bar\lambda^*)\zeta - \sum_{i=q+1}^{m} \frac{\mu}{c_i(x^*)} \nabla c_i(x^*)\\
+ O(\mu^{\sigma-1}) \sum_{i=1}^{q} \left(\bar\lambda_i^*\right)^2 \nabla c_i(x^*) + O(\mu^\sigma).
\end{aligned} \tag{27}$$

By using the definitions (6) and (7) and the decomposition (15), we have that

$$\begin{aligned}
U_N^T P_x(x;\mu) = \mu U_N^T \mathcal{L}_{xx}(x^*, \bar\lambda^*) U_N \zeta_N + \mu U_N^T \mathcal{L}_{xx}(x^*, \bar\lambda^*) U_R \zeta_R\\
- \mu \sum_{i=q+1}^{m} \frac{1}{c_i(x^*)} U_N^T \nabla c_i(x^*) + O(\mu^\sigma).
\end{aligned}$$

Hence by the definition (18) of ζ_N, we have that

$$U_N^T P_x(x;\mu) = O(\mu^\sigma). \tag{28}$$

Meanwhile, it follows immediately from (27) and the fact that $\sigma \in (1, 2]$ that

$$U_R^T P_x(x;\mu) = O(\mu^{\sigma-1}). \tag{29}$$

We now examine the Hessian $P_{xx}(x;\mu)$ for x satisfying the following bound:

$$\|x - (x^* + \mu\zeta)\| \le \rho\mu^\sigma. \tag{30}$$

By expanding in a similar fashion as for $P_x(\cdot;\mu)$, we obtain

$$\begin{aligned}
&P_{xx}(x;\mu) \\
&= \nabla^2 f(x^*) + O(\mu) - \sum_{i=1}^q \left[\nabla c_i(x^*)^T\zeta + O(\mu^{\sigma-1})\right]^{-1}\left[\nabla^2 c_i(x^*) + O(\mu)\right] \\
&\quad + \sum_{i=1}^q \mu^{-1}\left[\nabla c_i(x^*)^T\zeta + O(\mu^{\sigma-1})\right]^{-2}\left[\nabla c_i(x^*)\nabla c_i(x^*)^T\right. \\
&\qquad \left. + v_{1i}\nabla c_i(x^*)^T + \nabla c_i(x^*)v_{2i}^T + O(\mu^2)\right],
\end{aligned}$$

where v_{1i} and v_{2i}, $i = 1, 2, \ldots, q$ are vectors that satisfy the estimates

$$v_{1i} = O(\mu), \quad v_{2i} = O(\mu), \quad i = 1, 2, \ldots, q.$$

In expanding the last term, we have used the following relation, which holds for any two vectors a and b:

$$aa^T = bb^T + (a-b)b^T + b(a-b)^T + (a-b)(a-b)^T.$$

By using (14), we have that

$$\begin{aligned}
&P_{xx}(x;\mu) \\
&= \nabla^2 f(x^*) - \sum_{i=1}^q \bar\lambda_i^*\nabla^2 c_i(x^*) + \mu^{-1}\sum_{i=1}^q(\bar\lambda_i^*)^2\nabla c_i(x^*)\nabla c_i(x^*)^T \\
&\quad + O(\mu^{\sigma-2})\sum_{i=1}^q \nabla c_i(x^*)\nabla c_i(x^*)^T + \mu^{-1}\sum_{i=1}^q\left[\hat v_{1i}\nabla c_i(x^*)^T + \nabla c_i(x^*)\hat v_{2i}^T\right] + O(\mu),
\end{aligned} \tag{31}$$

where $\hat v_{1i}$ and $\hat v_{2i}$ are both of size $O(\mu)$ for all $i = 1, 2, \ldots, q$. We now examine the eigenstructure of this Hessian matrix. Using G to denote the active constraint gradients, that is,

$$G \triangleq \left[\nabla c_1(x^*), \nabla c_2(x^*), \ldots, \nabla c_q(x^*)\right],$$

with rank $G = \bar q \le q$, we have a $\bar q \times q$ matrix R with full row rank such that

$$G = U_R R.$$

Focusing on the $O(\mu^{-1})$ term in (31), we have that

$$\begin{aligned} P_{xx}(x;\mu) &= \mu^{-1} \sum_{i=1}^{q} (\bar{\lambda}_i^*)^2 \nabla c_i(x^*) \nabla c_i(x^*)^T + O(\mu^{\sigma-2}) \\ &= \mu^{-1} G D G^T + O(\mu^{\sigma-2}) \\ &= \mu^{-1} U_R (R D R^T) U_R^T + O(\mu^{\sigma-2}), \end{aligned} \tag{32}$$

where D is the diagonal matrix whose diagonal elements are $(\bar{\lambda}_i^*)^2$, $i = 1, 2, \dots, q$, all of which are positive. Denoting

$$H_{RR} = \mu^{-1} R D R^T,$$

we have by the properties of R and D that H_{RR} is a symmetric positive definite matrix, all of whose eigenvalues are of size $\Theta(\mu^{-1})$. In particular, we have

$$\|H_{RR}\| = O(\mu^{-1}), \quad \left\|H_{RR}^{-1}\right\| = O(\mu). \tag{33}$$

Therefore from (32), we have that

$$U_R^T P_{xx}(x;\mu) U_R = \mu^{-1} H_{RR} + O(\mu^{\sigma-2}). \tag{34}$$

We have by the definition of U_N together with (31) that

$$\begin{aligned} H_{NN} &\triangleq U_N^T P_{xx}(x;\mu) U_N \\ &= U_N^T \left[\nabla^2 f(x^*) - \sum_{i=1}^{q} \bar{\lambda}_i^* \nabla^2 c_i(x^*) \right] U_N + O(\mu) \\ &= U_N^T \mathcal{L}_{xx}(x^*, \bar{\lambda}^*) U_N + O(\mu), \end{aligned} \tag{35}$$

and by the second-order sufficient condition, we have that this matrix is positive-definite with all eigenvalues of size $\Theta(1)$, for all μ sufficiently small.

For the cross-term, we have that

$$H_{NR} \triangleq U_N^T P_{xx}(x;\mu) U_R = U_N^T \mathcal{L}_{xx}(x^*, \bar{\lambda}^*) U_R + \mu^{-1} U_N^T \hat{V}_1 G^T U_R + O(\mu),$$

where $\hat{V}_1 = [\hat{v}_{11}, \dots, \hat{v}_{1q}] = O(\mu)$. It follows from this estimate that

$$H_{RN}^T = H_{NR} = O(1). \tag{36}$$

From a standard result (see, for example, Wright [26, Lemma 2]), we have that

$$\begin{bmatrix} H_{RR} & H_{RN} \\ H_{RN}^T & H_{NN} \end{bmatrix}^{-1} = \begin{bmatrix} J_{11} & J_{12} \\ J_{12}^T & J_{22} \end{bmatrix}, \tag{37}$$

where

$$\begin{aligned}
J_{22} &= \left(H_{NN} - H_{RN}^T H_{RR}^{-1} H_{RN}\right)^{-1} = \left(H_{NN} + O(\mu)\right)^{-1} \\
&= O(1), \qquad\qquad (38a) \\
J_{11} &= H_{RR}^{-1} + H_{RR}^{-1} H_{RN} \left(H_{NN} - H_{RN}^T H_{RR}^{-1} H_{RN}\right)^{-1} H_{RN}^T H_{RR}^{-1} \\
&= O(\mu), \qquad\qquad (38b) \\
J_{12} &= -H_{RR}^{-1} H_{RN} \left(H_{NN} - H_{RN}^T H_{RR}^{-1} H_{RN}\right)^{-1} \\
&= O(\mu). \qquad\qquad (38c)
\end{aligned}$$

For the Newton step from x, we have

$$s = -P_{xx}(x;\mu)^{-1} P_x(x;\mu) = -\left[\, U_R \; U_N \,\right] \begin{bmatrix} H_{RR} & H_{RN} \\ H_{RN}^T & H_{NN} \end{bmatrix}^{-1} \begin{bmatrix} U_R^T P_x(x;\mu) \\ U_N^T P_x(x;\mu) \end{bmatrix},$$

so it follows from (28), (29), and (38) that

$$\begin{aligned}
U_R^T s &= -J_{11} U_R^T P_x(x;\mu) - J_{12} U_N^T P_x(x;\mu) = O(\mu) O(\mu^{\sigma-1}) + O(\mu) O(\mu^{\sigma}) = O(\mu^{\sigma}), \\
U_N^T s &= -J_{12}^T U_R^T P_x(x;\mu) - J_{22} U_N^T P_x(x;\mu) = O(\mu) O(\mu^{\sigma-1}) + O(1) O(\mu^{\sigma}) = O(\mu^{\sigma}).
\end{aligned}$$

We conclude that $s = O(\mu^{\sigma})$.

We now restore the explicit dependence of the constant in the $O(\cdot)$ term on the radius ρ_0, and summarize our results by writing

$$\|x - \bar{x}(\mu)\| \le \rho_0 \mu^{\sigma} \;\Rightarrow\; \|s\| \le C_2 \mu^{\sigma}. \qquad (39)$$

where $C_2 = C_2(\rho_0)$ is a positive homogeneous function of ρ_0. ■

The next lemma concerns the *first two* Newton steps for $P(\cdot;\mu)$ taken from a point x close to $\bar{x}(\mu)$. It derives a bound on the second step in terms of the first.

Lemma 5. *Suppose that Assumption 1 holds. Given $\sigma \in (1,2]$, there is a radius $\rho > 0$ such that for any $\rho_1 \in (0,\rho]$ the following property holds: There is a quantity $C_4 > 0$ depending on ρ_1 such that for all x with*

$$\|\tilde{x} - (x^* + \mu\zeta)\| = \|\tilde{x} - \bar{x}(\mu)\| \le \rho_1 \mu^{\sigma}, \qquad (40)$$

with $\mu \in (0,1]$, we have

$$\|\tilde{s}^+\| \le C_4 \mu^{-1} \|\tilde{s}\|^2.$$

where $\tilde{s}$ and $\tilde{s}^+$ are the first and second Newton steps, respectively, from the point $\tilde{x}$. The dependence of C_4 on ρ_1 is positively homogeneous.

Proof. We have from Lemma 4 that

$$\|\tilde{s}\| \le C_2(\rho_1) \mu^{\sigma}, \qquad (41)$$

where we indicate the dependence of C_2 on ρ_1 explicitly. We now seek a bound on $\|\tilde{s}^+\|$. As in the proof of Lemma 4, we use order notation, with the understanding that the constant represented by the $O(\cdot)$ term can be made arbitrarily small by decreasing ρ_1 appropriately.

By Taylor's theorem, we have

$$\begin{aligned}
&P_x(\tilde{x}+\tilde{s};\mu) \\
&= P_x(\tilde{x};\mu) + P_{xx}(\tilde{x};\mu)\tilde{s} + \int_0^1 \left[P_{xx}(\tilde{x}+\tau\tilde{s};\mu) - P_{xx}(\tilde{x};\mu)\right]\tilde{s}\,d\tau, \\
&= \int_0^1 \left[P_{xx}(\tilde{x}+\tau\tilde{s};\mu) - P_{xx}(\tilde{x};\mu)\right]\tilde{s}\,d\tau. \qquad (42)
\end{aligned}$$

Techniques similar to those in Wright [26, (39)] can be used to analyze this integrand. We obtain

$$\int_0^1 \left[P_{xx}(\tilde{x}+\tau\tilde{s};\mu) - P_{xx}(\tilde{x};\mu)\right]\tilde{s}\,d\tau = \sum_{i=1}^{q} O(\mu^{-2}\|\tilde{s}\|^2)\nabla c_i(x^*) + O(\mu^{-1}\|\tilde{s}\|^2),$$

so we conclude from (6), (7), and (42), that

$$U_R^T P_x(\tilde{x}+\tilde{s};\mu) = O(\mu^{-2}\|\tilde{s}\|^2), \qquad (43a)$$
$$U_N^T P_x(\tilde{x}+\tilde{s};\mu) = O(\mu^{-1}\|\tilde{s}\|^2). \qquad (43b)$$

By choosing ρ_1 small enough that

$$\rho_1 + 2C_2(\rho_1) \le \rho, \qquad (44)$$

we ensure that $\tilde{x}+\tilde{s}$ lies inside a neighborhood of the form (26), within which the bounds (38) on the component blocks of the inverse Hessian $P_{xx}(\tilde{x}+\tilde{s};\mu)$ apply, because

$$\|\tilde{x}+\tilde{s}-\bar{x}(\mu)\| \le \|\tilde{x}-\bar{x}(\mu)\| + \|\tilde{s}\| \le \rho_1\mu^\sigma + C_2(\rho_1)\mu^\sigma \le \rho\mu^\sigma.$$

(Note that the bounds in question did not depend on the specific choice of ρ_0 in Lemma 4, but only on the upper bound ρ.) Hence, by using these bounds together with (43) in the same fashion as in the argument that led to the estimate (39), we obtain that there is a positive homogeneous function $C_4(\cdot)$ such that

$$\|\tilde{x}-\bar{x}(\mu)\| \le \rho_1\mu^\sigma \;\Rightarrow\; \|\tilde{s}^+\| \le C_4(\rho_1)\mu^{-1}\|\tilde{s}\|^2. \qquad (45)$$

∎

We now prove our main result, which shows the existence of a minimizer $x(\mu)$ of $P(\cdot;\mu)$ close to $\bar{x}(\mu)$, and moreover proves convergence of Newton's method to this point when started from a neighborhood of $\bar{x}(\mu)$.

Theorem 1. *Suppose that Assumption 1 holds. Given $\sigma \in (1,2]$, there are values $\rho_2 > 0$ and $\bar{\mu} > 0$ such that for all x^0 with*

$$\|x^0 - (x^* + \mu\zeta)\| = \|x^0 - \bar{x}(\mu)\| \le \rho_2\mu^\sigma, \qquad (46)$$

with $\mu \in (0, \bar{\mu}]$, the sequence of (full) Newton steps generated from x^0 is well defined and converges to a strict local minimizer $x(\mu)$ of $P(\cdot;\mu)$. Moreover, we have that

$$\|x(\mu) - \bar{x}(\mu)\| = O(\mu^2) \tag{47}$$

and that $x(\mu)$ is the only local minimizer of $P(\cdot;\mu)$ in the neighborhood defined by (46).

Proof. Let us denote the first Newton step by s^0, the second by s^1, and so on. Given ρ defined as in Lemma 4, choose ρ_1 to satisfy (44). Now choose ρ_2 such that

$$\rho_2 + 2C_2(\rho_2) \le \rho_1. \tag{48}$$

Finally, choose $\bar{\mu} \in (0, 1]$ such that

$$C_4(\rho_1)C_2(\rho_1)\bar{\mu}^{\sigma-1} \le 1/2. \tag{49}$$

Given x^0 satisfying (46), we set $\tilde{x} = x^0$ and note that, because of (48), the condition $\|\tilde{x} - \bar{x}(\mu)\| \le \rho_1\mu^\sigma$ is certainly satisfied. By identifying s^0 with $\tilde{s}$ in Lemma 5, we obtain

$$\|s^0\| \le C_2(\rho_1)\mu^\sigma, \tag{50}$$

while from (45) and the identification $\tilde{s}^+ = s^1$, we have

$$\|s^1\| \le C_4(\rho_1)\mu^{-1}\|s^0\|^2. \tag{51}$$

By combining (50) and (51) and using (49) we have that

$$\|s^1\| \le C_4(\rho_1)C_2(\rho_1)\mu^{\sigma-1}\|s^0\| \le (1/2)\|s^1\|. \tag{52}$$

We now repeat the argument with a new value of $\tilde{x}$, namely, $\tilde{x} = x^0 + s^0$. Since from (39) and (48), we have

$$\|\tilde{x} - \bar{x}(\mu)\| \le \|x^0 - \bar{x}(\mu)\| + \|s^0\| \le [\rho_0 + C_2(\rho_0)]\,\mu^\sigma \le \rho_1\mu^\sigma,$$

it follows from (45) and the identification $\tilde{s}^+ = s^2$, $\tilde{s} = s^1$ that

$$\|s^2\| \le C_4(\rho_1)\mu^{-1}\|s^1\|^2. \tag{53}$$

Hence, from (52), (50), and (49), we have for $\mu \le \bar{\mu}$ that

$$\begin{aligned} \|s^2\| &\le \left[C_4(\rho_1)\mu^{-1}\|s^1\|\right]\|s^1\| \\ &\le (1/2)\left[C_4(\rho_1)\mu^{-1}\|s^0\|\right]\|s^1\| \le \left[C_4(\rho_1)\mu^{\sigma-1}\right]\|s^1\| \le (1/4)\|s^1\|. \end{aligned} \tag{54}$$

We continue in this fashion, setting $\tilde{x} = x^0+s^0+s^1$, $\tilde{x} = x^0+s^0+s^1+s^2$, and so on, to deduce that the Newton sequence is a Cauchy sequence and therefore is convergent to some point $x(\mu)$, where

$$\|x^0 - x(\mu)\| \le \left\|\sum_{i=0}^{\infty} s^i\right\| \le \sum_{i=0}^{\infty}\|s^i\| \le \sum_{i=0}^{\infty} 2^{-i}\|s^0\| = 2\|s^0\| \le 2C_2(\rho_1)\mu^\sigma. \tag{55}$$

Because of (44), we find that $x(\mu)$ lies within the region defined by (30), so that its Hessian $P_{xx}(x(\mu);\mu)$ is positive definite. Moreover, it follows from (43) by substituting successive values of $\tilde{x}$ and $\tilde{s}$ that $P_x(x(\mu);\mu) = 0$. Hence, $x(\mu)$ is a strict local minimizer of $P(\cdot;\mu)$. Since $P_{xx}(\cdot;\mu)$ remains positive definite for all x in the region defined by (30), we have in fact that $x(\mu)$ is the *unique* minimizer of $P(\cdot;\mu)$ in this neighborhood. Hence, because of (44) and (48), it is a fortiori the unique minimizer in the neighborhood (46), verifying the final statement of the theorem.

The Q-quadratic rate of convergence of the Newton sequence to $x(\mu)$ with rate constant of size $O(\mu^{-1})$ follows immediately from (45).

To verify the estimate (47), we simply consider the Newton sequence that starts at $x = \bar{x}(\mu)$. This starting point certainly satisfies (46) for $\sigma = 2$ (indeed, the left-hand side in this inequality is zero), so by applying our analysis with this value of σ, we obtain the result from (55). ∎

We now prove that the minimizers $x(\mu)$ described in Theorem 1 are unique in a certain sense.

Theorem 2. *Suppose that Assumption 1 holds. For all $\{\mu_k\}$ and $\{z^k\}$ with the properties that*

$$\mu_k \downarrow 0, \quad z^k \to x^*, \quad z^k \text{ a local min of } P(\cdot;\mu_k), \tag{56}$$

we have that $\mu_k/c_i(z^k) \to \bar{\lambda}_i^*$ for all $i = 1,2,\ldots,m$, and in fact that $z^k = x(\mu_k)$ for all k sufficiently large.

Proof. We first show that $\nabla c_i(x^*)^T(z^k - x^*) = o(\|z^k - x^*\|)$ cannot occur for any active index $i = 1,2,\ldots,q$.

We know from Lemma 3 that $\operatorname{dist}_{\mathcal{S}_\lambda} \lambda(z^k,\mu_k) \to 0$, so that all limit points of $\{\lambda(z^k,\mu_k)\}$ lie in $\mathcal{S}_\lambda$, by closedness. Let $\hat{\lambda}$ be any limit point of this sequence, and suppose WLOG that

$$\lambda(z^k,\mu_k) \to \hat{\lambda}. \tag{57}$$

By taking a further subsequence, we can assume that

$$\frac{z^k - x^*}{\|z^k - x^*\|} \to d, \tag{58}$$

for some d with $\|d\| = 1$, by compactness of the unit ball. Since $c_i(z^k) \geq 0$ and $c_i(x^*) = 0$ for each $i = 1,2,\ldots,q$, we have that

$$\nabla c_i(x^*)^T d \geq 0, \quad i = 1,2,\ldots,q. \tag{59}$$

By the definition (19) of $\lambda(z,\mu)$, we have by expanding $c_i(z^k)$ that

$$\begin{aligned} \frac{\mu_k}{c_i(z^k)} &= \frac{\mu_k}{\nabla c_i(x^*)^T d\|z^k - x^*\| + o(\|z^k - x^*\|)} \\ &\to \hat{\lambda}_i, \quad \text{for all } i = 1,2,\ldots,q. \end{aligned} \tag{60}$$

Let $\mathcal{Z}$ denote the following set of indices:

$$\mathcal{Z} \stackrel{\Delta}{=} \{j = 1,2,\ldots,q \mid \nabla c_j(x^*)^T d = 0\}, \tag{61}$$

and let $\mathcal{Z}^c$ denote $\{1, 2, \ldots, q\} \backslash \mathcal{Z}$. Our aim is to show that $\mathcal{Z} = \emptyset$. Suppose for contradiction that $\mathcal{Z}$ has at least one element. By taking an index $i \in \mathcal{Z}$ in (60), we have immediately that

$$\mu_k = o(\|z^k - x^*\|). \tag{62}$$

Hence, for the indices $i \in \mathcal{Z}^c$, those for which $\nabla c_i(x^*)^T d > 0$, we have again from (60) that $\hat{\lambda}_i = 0$. Therefore from the KKT condition (9a), we have that

$$\nabla f(x^*) = \sum_{i=1}^{q} \hat{\lambda}_i \nabla c_i(x^*) = \sum_{i \in \mathcal{Z}} \hat{\lambda}_i \nabla c_i(x^*). \tag{63}$$

Now by the strict complementarity assumption (11), there is a multiplier λ^* such that

$$\nabla f(x^*) = \sum_{i=1}^{q} \lambda_i^* \nabla c_i(x^*), \quad \lambda_i^* > 0, \quad \text{for all } i = 1, 2, \ldots, q. \tag{64}$$

By taking differences in (63) and (64), we have that

$$\sum_{i \in \mathcal{Z}} (\lambda_i^* - \hat{\lambda}_i) \nabla c_i(x^*) + \sum_{i \in \mathcal{Z}^c} \lambda_i^* \nabla c_i(x^*) = 0. \tag{65}$$

By taking the inner product with d, and using the definition of $\mathcal{Z}$, we have

$$\sum_{i \in \mathcal{Z}^c} \lambda_i^* \nabla c_i(x^*)^T d = 0. \tag{66}$$

Since $\lambda_i^* > 0$ and since $\nabla c_i(z^*)^T d \geq 0$ by (59), we must have $\nabla c_i(z^*)^T d = 0$, so that $\mathcal{Z}^c$ must be empty. We conclude that, for the particular subsequence satisfying (57) and (58), either $\mathcal{Z} = \emptyset$ or $\mathcal{Z} = \{1, 2, \ldots, q\}$. We now eliminate the latter case by supposing for contradiction that it happens. Since z^k is a local minimizer of $P(\cdot; \mu_k)$, we have from (20a) and (3) that

$$P_x(z^k; \mu_k) = \mathcal{L}_x(z^k, \lambda(z^k, \mu_k)) = 0. \tag{67}$$

Since $\hat{\lambda} \in \mathcal{S}_\lambda$, we have that

$$\mathcal{L}_x(x^*, \hat{\lambda}) = 0.$$

By taking differences of these two expressions, we have that

$$\begin{aligned}
0 &= \mathcal{L}_x(z^k, \lambda(z^k, \mu_k)) - \mathcal{L}_x(x^*, \hat{\lambda}) \\
&= \mathcal{L}_x(z^k, \lambda(z^k, \mu_k)) - \mathcal{L}_x(z^k, \hat{\lambda}) + \mathcal{L}_x(z^k, \hat{\lambda}) - \mathcal{L}_x(x^*, \hat{\lambda}) \\
&= -\sum_{i=1}^{q} \left[\lambda_i(z^k, \mu_k) - \hat{\lambda}_i \right] \nabla c_i(z^k) - \sum_{i=q+1}^{m} \lambda_i(z^k, \mu_k) \nabla c_i(z^k) \\
&\quad + \mathcal{L}_{xx}(x^*, \hat{\lambda})(z^k - x^*) + o(\|z^k - x^*\|) \\
&= \sum_{i=1}^{q} o(1) \nabla c_i(x^*) + o(\|z^k - x^*\|) + \mathcal{L}_{xx}(x^*, \hat{\lambda})(z^k - x^*) + O(\mu_k),
\end{aligned} \tag{68}$$

where we have used the estimate $\lambda_i(z^k, \mu_k) = O(\mu_k)$ for $i = q+1, \ldots, m$ to derive the final equality. Taking the inner product with d, and noting that $\nabla c_i(x^*)^T d = 0$ for all $i = 1, 2, \ldots, q$, we have that

$$0 = d^T \mathcal{L}_{xx}(x^*, \hat{\lambda})(z^k - x^*) + O(\mu_k) + o(\|z^k - x^*\|).$$

If we divide by $\|z^k - x^*\|$, take the limit, and use (62), we obtain

$$0 = d^T \mathcal{L}_{xx}(x^*, \hat{\lambda}) d.$$

However, the second-order conditions (16) require that $d^T \mathcal{L}_{xx}(x^*, \hat{\lambda}) d > 0$ for all $d \neq 0$ with $\nabla c_i(x^*)^T d = 0$, giving a contradiction. We conclude that $\mathcal{Z} = \emptyset$ for the direction d defined by the subsequence we have been considering—the one that satisfies (57) and (58). However, by compactness of $\mathcal{S}_\lambda$ and the unit ball, we can assign *every* index k to a subsequence that satisfies conditions of the type (57) and (58). Hence, we conclude that $\nabla c_i(x^*)^T d > 0$ for all possible approach directions d, or in other words that $\nabla c_i(x^*)^T (z^k - x^*) = o(\|z^k - x^*\|)$ cannot occur for any active index $i = 1, 2, \ldots, q$. Therefore, $\nabla c_i(x^*)^T d > 0$ for all $i = 1, 2, \ldots, q$.

By taking a further subsequence in (60), we can assume that

$$\frac{\|z^k - x^*\|}{\mu_k} \to \gamma. \tag{69}$$

By (60) and finiteness of $\hat{\lambda}$, we have $\gamma > 0$. We also have $\gamma < \infty$, since otherwise the limit in (60) would be $\hat{\lambda} = 0$, so that $\alpha \lambda^* \in \mathcal{S}_\lambda$ for all $\alpha \geq 0$ and all strictly complementary $\lambda^* \in \mathcal{S}_\lambda$, contradicting boundedness of $\mathcal{S}_\lambda$. Hence, from (60), we have

$$\frac{1}{\nabla c_i(x^*)^T (\gamma d)} = \hat{\lambda}_i, \quad i = 1, 2, \ldots, q. \tag{70}$$

It follows from the optimality conditions for the analytic center problem (12), (13) and the uniqueness of the solution $\bar{\lambda}^*$ of this problem that

$$\hat{\lambda} = \bar{\lambda}^*. \tag{71}$$

Since *all* limit points of $\lambda(z^k, \mu_k)$ have the property (71), we conclude that $\lambda(z^k, \mu_k) \to \bar{\lambda}^*$.

The remainder of the proof establishes $z_k = x(\mu_k)$ by showing that if these two local minimizers of $P(\cdot; \mu_k)$ are distinct, the Hessian $P_{xx}(\cdot; \mu)$ is positive definite at all points along the straight line that connects them.

Since by (70) and (71), the projection of (γd) onto the subspace

$$\text{Range}\,\{\nabla c_i(x^*) \mid i = 1, 2, \ldots, q\}$$

is uniquely defined, we have that, for the entire sequence $\{z^k\}$ satisfying (56),

$$\lim_k \nabla c_i(x^*)^T \frac{z^k - x^*}{\|z^k - x^*\|} = \nabla c_i(x^*)^T d = \nabla c_i(x^*)^T \frac{\zeta}{\|\zeta\|}, \quad i = 1, 2, \ldots, q, \tag{72}$$

and

$$\gamma = \|\zeta\|, \tag{73}$$

where ζ is the vector defined in (14), (15), and (18). It follows from these observations together with (14), (25), (69), and Theorem 1 that

$$\begin{aligned}
\nabla c_i(x^*)^T(z^k - x^*) &= \gamma^{-1}\|z^k - x^*\|\nabla c_i(x^*)^T\zeta + o(\mu_k) \\
&= \mu_k \nabla c_i(x^*)^T\zeta + o(\mu_k) \\
&= \nabla c_i(x^*)^T\left(\bar{x}(\mu_k) - x^*\right) + o(\mu_k) \\
&= \nabla c_i(x^*)^T\left(x(\mu_k) - x^*\right) + o(\mu_k), \quad i = 1, 2, \ldots, q,
\end{aligned}$$

and therefore

$$\nabla c_i(x^*)^T\left(x(\mu_k) - z^k\right) = o(\mu_k), \quad i = 1, 2, \ldots, q.$$

Hence, we have for all $\alpha \in [0, 1]$ that

$$\begin{aligned}
&c_i\left(z^k + \alpha(x(\mu_k) - z^k)\right) \\
&= c_i(z^k) + \alpha\nabla c_i(z^k)^T(x(\mu_k) - z^k) + O(\mu_k^2) \\
&= \left[\mu_k/\bar{\lambda}_i^* + o(\mu_k)\right] + \alpha\nabla c_i(x^*)^T(x(\mu_k) - z^k) + O(\mu_k^2) \\
&= \mu_k/\bar{\lambda}_i^* + o(\mu_k), \qquad i = 1, 2, \ldots, q.
\end{aligned} \tag{74}$$

We now consider the Hessian $P_{xx}(\cdot;\mu_k)$ evaluated at the points $z^k + \alpha(x(\mu_k) - z^k)$, $\alpha \in [0, 1]$. By using analysis very similar to that in the proof of Lemma 4, together with the observation (74), we can show that this matrix is positive definite for all $\alpha \in [0, 1]$, for all k sufficiently large. By Taylor's theorem, we have

$$\begin{aligned}
0 &= \left(z^k - x(\mu_k)\right)^T\left[P_x(z^k;\mu_k) - P_x(x(\mu_k);\mu_k)\right] \\
&= \int_0^1 \left(z^k - x(\mu_k)\right)^T P_{xx}\left(z^k + \alpha(x(\mu_k) - z^k)\right)\left(z^k - x(\mu_k)\right)\, d\alpha.
\end{aligned}$$

Observe that the right-hand side of this expression is positive whenever $z^k \neq x(\mu_k)$ for all k sufficiently large. We conclude that $z^k = x(\mu_k)$ for all k sufficiently large, as required. ∎

We now demonstrate the differentiability of the path $x(\mu)$ discussed in Theorems 1 and 2. The proof again uses the decomposition of the Hessian $P_{xx}(x;\mu)$ that was first derived in the proof of Lemma 4.

Theorem 3. *Suppose that Assumption 1 holds. Then for the minimizers $x(\mu)$ defined in Theorem 1, there is a threshold $\bar{\mu}$ such $x(\mu)$ exists and is a continuously differentiable function of μ for all $\mu \in (0, \bar{\mu}]$, and we have that*

$$\lim_{\mu \downarrow 0} \dot{x}(\mu) = \zeta. \tag{75}$$

Proof. Choose $\sigma = 2$ in Theorem 1, and let $\bar{\mu}$ and C_0 be defined accordingly. We now have existence of $x(\mu)$ for all $\mu \in (0, \bar{\mu}]$, and the estimate (47) holds. Each such $x(\mu)$ solves the equation

$$P_x(x(\mu); \mu) = 0. \tag{76}$$

P_{xx} is nonsingular and continuous at each $\mu \in (0, \bar{\mu}]$, by (37), (38) and the discussion preceding these expressions. Hence, we can apply the implicit function theorem (see, for example, Ortega and Rheinboldt [19, p. 128]) to conclude that $x(\cdot)$ is differentiable at μ and that the derivative $\dot{x}(\mu)$ satisfies the equation

$$P_{xx}(x(\mu), \mu)\dot{x}(\mu) - r(\mu) = 0, \tag{77}$$

where

$$r(\mu) \stackrel{\triangle}{=} \sum_{i=1}^{m} \frac{1}{c_i(x(\mu))} \nabla c_i(x(\mu)). \tag{78}$$

We now show (75) by showing that $\dot{x}(\mu) = \zeta + u(\mu)$, where $u(\mu) = O(\mu)$ and so, in particular, $u(\mu) \to 0$ as $\mu \downarrow 0$. By substituting into (77), we have that u satisfies the expression

$$P_{xx}(x(\mu), \mu)u = r(\mu) - P_{xx}(x(\mu), \mu)\zeta. \tag{79}$$

We can substitute $x(\mu)$ for x in the analysis of $P_{xx}(\cdot; \mu)$ that leads to (37), (38), since $x(\mu)$ certainly lies in the neighborhood (46) within which this estimate is valid. Hence, by applying (37) to (79), we have that

$$\begin{bmatrix} U_R^T u \\ U_N^T u \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{12}^T & J_{22} \end{bmatrix} \begin{bmatrix} U_R^T \\ U_N^T \end{bmatrix} \left[r(\mu) - P_{xx}(x(\mu), \mu)\zeta \right]. \tag{80}$$

Hence, from the estimates (38), we have that

$$u(\mu) = O(\mu) \left\| U_R^T [r(\mu) - P_{xx}(x(\mu), \mu)\zeta] \right\| + O(1) \left\| U_N^T [r(\mu) - P_{xx}(x(\mu), \mu)\zeta] \right\|.$$

Therefore, our estimate $u(\mu) = O(\mu)$ will follow if we can show that

$$U_R^T [r(\mu) - P_{xx}(x(\mu), \mu)\zeta] = O(1), \tag{81a}$$

$$U_N^T [r(\mu) - P_{xx}(x(\mu), \mu)\zeta] = O(\mu). \tag{81b}$$

By substituting directly from (20b) and (78), we have that

$$\begin{aligned} P_{xx}(x(\mu); \mu)\zeta - r(\mu) = {} & \left[\nabla^2 f(x(\mu)) + \sum_{i=1}^{m} \frac{\mu}{c_i(x(\mu))} \nabla^2 c_i(x(\mu)) \right] \zeta \\ & + \sum_{i=1}^{q} \frac{1}{c_i(x(\mu))} \left[\frac{\mu \nabla c_i(x(\mu))^T \zeta}{c_i(x(\mu))} - 1 \right] \nabla c_i(x(\mu)) \\ & + \sum_{i=q+1}^{m} \frac{1}{c_i(x(\mu))} \left[\frac{\mu \nabla c_i(x(\mu))^T \zeta}{c_i(x(\mu))} - 1 \right] \nabla c_i(x(\mu)). \end{aligned} \tag{82}$$

By using (14) together with the estimates (46) (with $\sigma = 2$) and (47), we obtain that

$$\begin{aligned} c_i(x(\mu)) &= \nabla c_i(x^*)^T (x(\mu) - x^*) + O(\mu^2) \\ &= \mu \nabla c_i(x^*)^T \zeta + O(\mu^2) = \frac{\mu}{\bar{\lambda}_i^*} + O(\mu^2), \quad i = 1, 2, \ldots, q. \end{aligned} \tag{83}$$

Hence, Lipschitz continuity of $\nabla^2 f(\cdot)$ and $\nabla^2 c_i(\cdot)$, $i = 1, 2, \ldots, m$ implies that

$$\nabla^2 f(x(\mu)) + \sum_{i=1}^{m} \frac{\mu}{c_i(x(\mu))} \nabla^2 c_i(x(\mu)) = \nabla^2 f(x^*) + \sum_{i=1}^{m} \bar{\lambda}_i^* \nabla^2 c_i(x^*) + O(\mu). \tag{84}$$

For the second term on the right-hand side of (82), we use (83) again to obtain

$$\frac{\mu \nabla c_i(x(\mu))^T \zeta}{c_i(x(\mu))} - 1 = \frac{\mu/\bar{\lambda}_i^* + O(\mu^2)}{\mu/\bar{\lambda}_i^* + O(\mu^2)} - 1 = O(\mu), \quad i = 1, 2, \ldots, q. \tag{85}$$

Hence, by using (83) again together with the property $\bar{\lambda}_i^* > 0$, $i = 1, 2, \ldots, q$, we have that

$$\begin{aligned} &\sum_{i=1}^{q} \frac{1}{c_i(x(\mu))} \left[\frac{\mu \nabla c_i(x(\mu))^T \zeta}{c_i(x(\mu))} - 1 \right] \nabla c_i(x(\mu)) \\ &= \sum_{i=1}^{q} \frac{O(\mu)}{c_i(x(\mu))} \nabla c_i(x(\mu)) = \sum_{i=1}^{q} O(1) \nabla c_i(x^*) + O(\mu). \end{aligned} \tag{86}$$

For the third term on the right-hand side of (82), we have that, since $c_i(x(\mu))$ is bounded away from zero for μ sufficiently small,

$$\sum_{i=q+1}^{m} \frac{1}{c_i(x(\mu))} \left[\frac{\mu \nabla c_i(x(\mu))^T \zeta}{c_i(x(\mu))} - 1 \right] \nabla c_i(x(\mu)) = - \sum_{i=q+1}^{m} \frac{1}{c_i(x^*)} \nabla c_i(x^*) + O(\mu). \tag{87}$$

By substituting (84), (86), (87) into (82), and taking the inner product with U_R^T, we have that (81a) is satisfied. When we take the inner product with U_N^T, the terms involving $\nabla c_i(x^*)$, $i = 1, 2, \ldots, q$ in (86) are eliminated, and we are left with

$$U_N^T [P_{xx}(x(\mu); \mu)\zeta - r(\mu)] = U_N^T \mathcal{L}_{xx}(x^*, \bar{\lambda}^*)\zeta - \sum_{i=q+1}^{m} \frac{1}{c_i(x^*)} U_N^T \nabla c_i(x^*) + O(\mu). \tag{88}$$

By comparing this expression with (18), we conclude that (81b) is satisfied, completing the proof. ∎

The relation (75) together with (14) shows that the primal central path reaches x^* nontangentially to the active constraints, since the strict complementarity is certainly satisfied at $\bar{\lambda}^*$, but it is tangential to the linear path $\{\bar{x}(\mu) \mid \mu \in (0, \overline{\mu}]\}$ at x^*. It also shows that $p = -\zeta = -\dot{x}(0)$ satisfies the MFCQ (8).

The proof of (75) is much simpler in the case of linearly independent active constraints. When this condition holds, Fiacco and McCormick [8, Section 5.2] replace (77) by an "augmented" linear system whose unknowns are both $\dot{x}(\mu)$ and $\dot{\lambda}(\mu)$ and whose coefficient matrix approaches a nonsingular limit as $\mu \downarrow 0$. The result follows by setting $\mu = 0$ and calculating the solution of this system directly. M. H. Wright performs a similar analysis [22, Section 3] and observes the nontangentiality of the path to the active constraints.

4. Relaxing the Strict Complementarity Condition

In this section, we discuss the properties of the sequence of minimizers of $P(\cdot;\mu_k)$ when strict complementarity (11) does not hold. That is, we have for some active constraint index $i = 1, 2, \ldots, q$ that $\lambda_i^* = 0$ for all $\lambda^* \in \mathcal{S}_\lambda$. Lemmas 1, 2, and 3 continue to hold when (11) is not satisfied. However, the problem (12), (13) that defines the analytic center is not even feasible, so the path of minimizers of $P(\cdot;\mu)$ with the particular form described in Section 3 is not defined.

Our main results are as follows. Under a suitably modified second-order sufficient condition, we can show existence of a local minimizer of $P(\cdot;\mu_k)$ in the vicinity of x^*, for all μ_k sufficiently small, using a simple modification of results from [22]. We show that the direction of approach of the sequence of minimizers to x^* is tangential to the strongly active constraints (those for which $\lambda_i^* > 0$ for some $\lambda^* \in \mathcal{S}_\lambda$). Finally, we show that

$$\mu_k = \Theta(\|z^k - x^*\|^2),$$

where z^k is the local minimizer of $P(\cdot;\mu_k)$. This contrasts with the strictly complementary case, in which the exponent 2 does not appear. We are not able to prove local uniqueness of the minimizer (as was seen for the strictly complementary case in Theorem 2), nor are we able to obtain the semi-explicit characterization of the minimizer seen in Theorem 1.

We suppose that the "non-strictly complementary" indices are $\bar{q}+1, \ldots, q$, that is,

$$\lambda_i^* = 0, \quad \text{for all } \lambda^* \in \mathcal{S}_\lambda, \text{ all } i = \bar{q}+1, \ldots, q. \tag{89}$$

We modify the second-order conditions (16) as follows:

$$\begin{gathered} d^T \mathcal{L}_{xx}(x^*, \lambda^*) d > 0, \quad \text{for all } \lambda^* \in \mathcal{S}_\lambda \\ \text{and all } d \neq 0 \text{ with } \nabla c_i(x^*)^T d = 0 \text{ for all } i = 1, 2, \ldots, \bar{q} \\ \text{and } \nabla c_i(x^*)^T d \geq 0 \text{ for all } i = \bar{q}+1, \ldots, q. \end{gathered} \tag{90}$$

Under these conditions, x^* remains a strict local solution of (1).

We can obtain some insight into this case by considering the following simple example:

$$\min \tfrac{1}{2}\left(x_1^2 + x_2^2\right) \quad \text{subject to } x_1 \geq 1,\ x_2 \geq 0. \tag{91}$$

The solution is $x^* = (1,0)^T$, with both constraints active and unique optimal Lagrange multipliers $\lambda_1^* = 1$, $\lambda_2^* = 0$. It is easy to verify that the minimizer of $P(\cdot;\mu)$ in this case is

$$x(\mu) = \left(\frac{1+\sqrt{1+4\mu}}{2}, \sqrt{\mu} \right)^T \approx (1+\mu, \sqrt{\mu})^T .$$

The path of minimizers is dramatically different from the one that would be obtained by omitting the weakly active constraint $x_2 \geq 0$ from the problem, which would be

$$x(\mu) = \left(\frac{1+\sqrt{1+4\mu}}{2}, 0 \right)^T \approx (1+\mu, 0)^T .$$

Note that the path becomes tangential to the strongly active constraint $x_1 \geq 1$ and that the distance from $x(\mu)$ to the solution x^* is $O(\mu^{1/2})$ rather than $O(\mu)$, as in the case of strict complementarity.

All results in this section use the following assumption.

Assumption 2. *The first-order necessary conditions (4), the second-order sufficient conditions (90), and the MFCQ (8) hold at x^*. The strict complementarity condition fails to hold, that is, $\bar{q} < q$ in (89).*

We start with a result on the existence of a sequence of minimizers of the barrier function that approaches x^*. It is a consequence of Theorem 7 in M. H. Wright [22]. Under our assumptions, x^* is a strict local minimizer of the problem (1), and so the set $\mathcal{M}$ in the cited result is the singleton $\{x^*\}$.

Theorem 4. *Suppose that Assumption 2 holds. Let $\{\mu_k\}$ be any sequence of positive numbers such that $\mu_k \downarrow 0$. Then*

(i) there exists a neighborhood $\mathcal{N}$ of x^* such that for all k sufficiently large, $P(\cdot;\mu_k)$ has at least one unconstrained minimizer in strict $\mathcal{C} \cap \mathcal{N}$. *Moreover, every sequence of* global *minimizers $\{\bar{x}_k\}$ of $P(\cdot;\mu_k)$ in* strict $\mathcal{C} \cap \mathrm{cl}\,\mathcal{N}$ *converges to x^*.*
(ii) $\lim_{k\to\infty} f(\bar{x}_k) = \lim_{k\to\infty} P(\bar{x}_k;\mu_k) = f(x^*)$.

The next three results concern the behavior of any sequences $\{\mu_k\}$ and $\{z^k\}$ with the following properties:

$$\mu_k \downarrow 0, \quad z^k \to x^*, \quad z^k \text{ a local min of } P(\cdot;\mu_k). \tag{92}$$

The sequence of global minimizers $\{\bar{x}_k\}$ described in Theorem 4 is one possible choice for $\{z^k\}$.

Theorem 5. *Suppose that Assumption 2 holds. Let $\{\mu_k\}$ and $\{z^k\}$ be sequences with the properties (92). Then we have that*

$$\mu_k = O(\|z^k - x^*\|^2). \tag{93}$$

Proof. From Lemma 3, we have that

$$\lambda_i(z^k,\mu_k) \le \text{dist}_{\mathcal{S}_\lambda} \lambda(z^k;\mu_k) = O\left(\|z^k - x^*\| + \mu_k\right), \quad \text{for all } i = \bar{q}+1,\ldots,q.$$

By substituting from (19), and using the estimate $c_i(z^k) = O\left(\|z^k - x^*\|\right)$, we have that

$$\mu_k \le c_i(z^k) O\left(\|z^k - x^*\| + \mu_k\right) \le K_1\left(\|z^k - x^*\|^2 + \mu_k\|z^k - x^*\|\right),$$

for some $K_1 > 0$ and all k sufficiently large. Therefore, we have

$$\left(1 - K_1\|z^k - x^*\|\right)\mu_k \le K_1\|z^k - x^*\|^2,$$

so by taking k large enough that $\|z^k - x^*\| \le 1/(2K_1)$, we have the result. ∎

We now show that the approach of the minimizer sequence is tangential to the strongly active constraints.

Lemma 6. *Suppose that Assumption 2 holds. Let $\{\mu_k\}$ and $\{z^k\}$ be sequences such that*

$$\mu_k \downarrow 0, \quad z^k \to x^*, \quad z^k \text{ a local min of } P(\cdot;\mu_k), \quad \frac{z^k - x^*}{\|z^k - x^*\|} \to d, \tag{94}$$

for some vector $d \in \mathrm{R}^n$ with $\|d\| = 1$.

$$\nabla c_i(x^*)^T d = 0, \quad \text{for all } i = 1,2,\ldots,\bar{q}. \tag{95}$$

Proof. Given the sequences $\{\mu_k\}$ and $\{z^k\}$ satisfying (94), let $\hat{\lambda}$ be a limit point in $\mathcal{S}_\lambda$ of the sequence $\{\lambda(z^k,\mu_k)\}$. We know that $\hat{\lambda}_i = 0$ for all $i = \bar{q}+1,\ldots,q$. Given our definition of $\bar{q}$ and convexity of the set $\mathcal{S}_\lambda$, we can choose a multiplier $\lambda^* \in \mathcal{S}_\lambda$ such that

$$\lambda_i^* > 0, \quad \text{for all } i = 1,2,\ldots,\bar{q}.$$

We cannot have $\hat{\lambda} = 0$, since it would then follow from (9a) that $\nabla f(x^*) = 0$ and therefore that $\alpha\lambda^* \in \mathcal{S}_\lambda$ for all $\alpha \ge 0$. Hence $\mathcal{S}_\lambda$ would be unbounded, contradicting (8) which, as we noted earlier, ensures boundedness of $\mathcal{S}_\lambda$. Therefore $\hat{\lambda}_i > 0$ for at least one $i = 1,2,\ldots,\bar{q}$.

Let $\mathcal{K}$ be a subsequence of indices such that $\lim_{k\in\mathcal{K}} \lambda(z^k,\mu_k) = \hat{\lambda}$. For the index i such that $\hat{\lambda}_i > 0$, we have

$$\frac{\mu_k}{c_i(z^k)} \ge \hat{\lambda}_i/2, \quad \text{for all } k \in \mathcal{K} \text{ sufficiently large.}$$

Since $c_i(z_k)/\|z_k - x^*\| = \nabla c_i(x^*)^T d + o(1)$, we have by dividing numerator and denominator by $\|z^k - x^*\|$ that

$$\frac{\mu_k/\|z^k - x^*\|}{\nabla c_i(x^*)^T d + o(1)} \ge \hat{\lambda}_i/2, \quad \text{for all } k \in \mathcal{K} \text{ sufficiently large.}$$

If $\nabla c_i(x^*)^T d > 0$, we would have from (93) that the left-hand side in this expression approaches zero, a contradiction. Therefore, we have that $\nabla c_i(x^*)^T d = 0$ for all indices i such that $\hat{\lambda}_i > 0$.

Since $\lambda^* \in \mathcal{S}_\lambda$ and $\hat{\lambda} \in \mathcal{S}_\lambda$, we have that

$$\sum_{i=1}^{\bar{q}} \lambda_i^* \nabla c_i(x^*) = \sum_{i=1}^{\bar{q}} \hat{\lambda}_i \nabla c_i(x^*) = \sum_{\hat{\lambda}_i > 0} \hat{\lambda}_i \nabla c_i(x^*).$$

By rearranging this expression, we obtain

$$\sum_{\hat{\lambda}_i > 0} (\lambda_i^* - \hat{\lambda}_i) \nabla c_i(x^*) = - \sum_{\hat{\lambda}_i = 0} \lambda_i^* \nabla c_i(x^*).$$

By taking the inner product of both sides with d, and using that $d^T \nabla c_i(x^*) = 0$ for all i with $\hat{\lambda}_i > 0$, we obtain that

$$\sum_{\hat{\lambda}_i = 0} \lambda_i^* d^T \nabla c_i(x^*) = 0.$$

Since $\lambda_i^* > 0$ for all i in this sum, and since feasibility dictates that $\nabla c_i(x^*)^T d \geq 0$, we conclude that $\nabla c_i(x^*)^T d = 0$ for the indices $i = 1, 2, \ldots, \bar{q}$ with $\hat{\lambda}_i = 0$ as well. Hence, we have proved (95). ∎

We now show that there is a lower bound on μ_k in terms of $\|z^k - x^*\|^2$ to go with the upper bound in (93).

Theorem 6. *Suppose that Assumption 2 holds. Let $\{\mu_k\}$ and $\{z^k\}$ be sequences with the properties (92). Then we have that*

$$\|z^k - x^*\|^2 = O(\mu_k). \tag{96}$$

Proof. Suppose that d is a limiting direction and $\hat{\lambda}$ is a limiting multiplier of the sequence (92), that is, for some subsequence $\mathcal{K}$, we have

$$\frac{z^k - x^*}{\|z^k - x^*\|} \to_{\mathcal{K}} d, \quad \lambda(z^k, \mu_k) \to_{\mathcal{K}} \hat{\lambda} \in \mathcal{S}_\lambda. \tag{97}$$

Since

$$\mathcal{L}_x(z^k, \lambda(z^k, \mu_k)) = 0, \quad \mathcal{L}_x(x^*, \hat{\lambda}) = 0,$$

we have by expanding as in (68) that

$$\begin{aligned} 0 &= \mathcal{L}_x(z^k, \lambda(z^k, \mu_k)) - \mathcal{L}_x(x^*, \hat{\lambda}) \\ &= \mathcal{L}_x(z^k, \lambda(z^k, \mu_k)) - \mathcal{L}_x(z^k, \hat{\lambda}) + \mathcal{L}_x(z^k, \hat{\lambda}) - \mathcal{L}_x(x^*, \hat{\lambda}) \\ &= - \sum_{i=1}^{\bar{q}} \left[\lambda_i(z^k, \mu_k) - \hat{\lambda}_i \right] \nabla c_i(z^k) - \sum_{i=\bar{q}+1}^{q} \lambda_i(z^k, \mu_k) \nabla c_i(z^k) - \\ &\quad \sum_{i=q+1}^{m} \lambda_i(z^k, \mu_k) \nabla c_i(z^k) + \mathcal{L}_{xx}(x^*, \hat{\lambda})(z^k - x^*) + o(\|z^k - x^*\|) \end{aligned}$$

$$= \sum_{i=1}^{\bar{q}} o(1)\nabla c_i(x^*) - \sum_{i=\bar{q}+1}^{q} \lambda_i(z^k,\mu_k)\nabla c_i(x^*) \tag{98}$$
$$+\mathcal{L}_{xx}(x^*,\hat{\lambda})(z^k - x^*) + o(\|z^k - x^*\|) + O(\mu_k),$$

where we have used the following estimates:

$$\lambda_i(z^k,\mu_k) = O(\mu_k), \quad i = q+1,\ldots,m, \tag{99a}$$
$$\lambda_i(z^k,\mu_k) = O(\|z^k - x^*\|), \quad i = \bar{q}+1,\ldots,q, \tag{99b}$$
$$\lambda_i(z^k,\mu_k) - \hat{\lambda}_i = O(\|z^k - x^*\|), \quad i = 1,\ldots,\bar{q}, \tag{99c}$$
$$\nabla c_i(z^k) = \nabla c_i(x^*) + O(\|z^k - x^*\|), \quad i = 1,2,\ldots,m, \tag{99d}$$
$$z^k - x^* = d\|z^k - x^*\| + o(\|z^k - x^*\|). \tag{99e}$$

By taking inner products with $d/\|z^k - x^*\|$, and using (95) and (93), we have that

$$d^T\mathcal{L}_{xx}(z^*,\hat{\lambda})d = \sum_{i=\bar{q}+1}^{q} \frac{\lambda_i(z^k,\mu_k)}{\|z^k - x^*\|}\nabla c_i(x^*)^T d + o(1). \tag{100}$$

For $i = \bar{q}+1,\ldots,q$, we have that $\nabla c_i(x^*)^T d \geq 0$, since otherwise we would have $c_i(z^k) < 0$ for k sufficiently large, contradicting feasibility. Hence, using (95) again, we certainly have that

$$\nabla c_i(x^*)^T d = 0, \quad \text{all } i \text{ with } \hat{\lambda}_i > 0,$$
$$\nabla c_i(x^*)^T d \geq 0, \quad \text{all } i \text{ with } \hat{\lambda}_i = 0,$$

so from (90) we have that $d^T\mathcal{L}_{xx}(x^*,\hat{\lambda})d > 0$. Hence, the right-hand side in (100) must be bounded away from zero. Since each term in the summation is nonnegative, we must have for at least one $i = \bar{q}+1,\ldots,q$ that $\nabla c_i(x^*)^T d > 0$. For at least one such i, the corresponding coefficient in (100) must be positive in the limit, that is, there is a $\gamma > 0$ such that

$$\frac{\lambda_i(z^k,\mu_k)}{\|z^k - x^*\|} \geq \gamma, \quad \text{all } k, \text{ and at least one } i \text{ with } \nabla c_i(x^*)^T d > 0.$$

By expanding this coefficient, we obtain

$$\begin{aligned}\frac{\lambda_i(z^k,\mu_k)}{\|z^k - x^*\|} &= \frac{\mu_k}{c_i(z^k)\|z^k - x^*\|} \\ &= \frac{\mu_k}{\nabla c_i(x^*)^T d\|z^k - x^*\|^2 + o(\|z^k - x^*\|^2)} \\ &\geq \gamma.\end{aligned}$$

Therefore, we have

$$\mu_k \geq \gamma\nabla c_i(x^*)^T d\|z^k - x^*\|^2 + o(\|z^k - x^*\|^2) \geq (\gamma/2)\nabla c_i(x^*)^T d\|z^k - x^*\|^2,$$

for all k sufficiently large. Therefore, by taking

$$K_2 = \max_{i \mid \nabla c_i(x^*)^T d > 0} \frac{2}{\gamma \nabla c_i(x^*)^T d},$$

we have that $\|z^k - x^*\|^2 \le K_2 \mu_k$ for all k sufficiently large.

Hence, we have proved the result for the subsequence $\mathcal{K}$ with the properties (97). Since each index k can be assigned to a subsequence of this type, we conclude that the estimate $\|z^k - x^*\|^2 = O(\mu_k)$ holds for the entire sequence. ∎

This result appears similar to one of Mifflin [15, Theorem 5.4], but the assumptions on $\mathcal{L}(\cdot;\lambda^*)$ in that paper are strong. Essentially, they are that $\mathcal{L}(\cdot;\lambda^*)$ satisfy a strong convexity property over some convex set containing the iterates z^k, for all k sufficiently large.

The following result follows immediately from Theorems 5 and 6.

Corollary 1. *Suppose that Assumption 2 holds. Then any sequences* $\{\mu_k\}$ *and* $\{z^k\}$ *with the properties that*

$$\mu_k \downarrow 0, \quad z^k \to x^*, \quad z^k \ \ \textit{a local min of } P(\cdot;\mu_k), \tag{101}$$

will satisfy

$$\mu_k = \Theta(\|z^k - x^*\|^2). \tag{102}$$

5. Discussion

Motivated by the success of primal-dual interior-point methods on linear programming problems, a number of researchers recently have described primal-dual methods for nonlinear programming. In these methods, the Lagrange multipliers λ generally are treated as independent variables, rather than being defined in terms of the primal variables x by a formula such as (19). We mention in particular the work of Forsgren and Gill [9], El Bakry et al. [3], and Gay, Overton, and Wright [11], who use line-search methods, and Conn et al. [6] and Byrd, Gilbert, and Nocedal [5], who describe trust-region methods. Methods for nonlinear convex programming are described by Ralph and Wright [21, 20], among others.

Near the solution x^*, primal-dual methods gravitate toward points on the primal-dual central path, which is parametrized by μ and defined as the set of points $(x(\mu), \lambda(\mu))$ that satisfies the conditions

$$\nabla f(x) - \sum_{i=1}^{m} \lambda_i \nabla c_i(x) = 0, \tag{103a}$$

$$\lambda_i c_i(x) = \mu, \ \text{ for all } i = 1, 2, \ldots, m, \tag{103b}$$

$$\lambda > 0, \quad c(x) > 0. \tag{103c}$$

When the LICQ and second-order sufficient conditions hold, the Jacobian matrix of the nonlinear equations formed by (103a) and (103b) is nonsingular in a neighborhood of (x^*, λ^*), where λ^* is the (unique) optimal multiplier. Fiacco and McCormick use this observation to differentiate the equations (103a) and (103b) with respect to μ, and thereby prove results about the smoothness of the trajectory $(x(\mu), \lambda(\mu))$ near (x^*, λ^*).

The results of Section 3 above show that the system (103) continues to have a solution in the neighborhood of $\{x^*\} \times \mathcal{S}_\lambda$ when LICQ is replaced by MFCQ. We simply take $x(\mu)$ to be the vector described in Theorems 1, 2, and 3, and define $\lambda(\mu)$ by (19). Hence, we have existence and local uniqueness of a solution even though the limiting Jacobian of (103a), (103b) is singular, and we find that the primal-dual trajectory approaches the specific limit point $(x^*, \bar{\lambda}^*)$. The smoothness properties of the path under MFCQ are not obvious, however.

In the case of no strict complementarity, the (weaker) existence results of Section 4 can again be used to deduce the existence of solutions to (103) near $\{x^*\} \times \mathcal{S}_\lambda$, but we cannot say much else about this case other than that the convergence rate implied by Lemma 3 is satisfied.

Finally, we comment about the use of Newton's method to minimize $P(\cdot;\mu)$ approximately for a decreasing sequence of values of μ, a scheme known as the Newton/log-barrier approach. Extrapolation can be used to obtain a starting point for the Newton iteration after each decrease in μ. Superlinear convergence of this approach is obtained by decreasing μ_k superlinearly to zero (that is, $\lim_{k\to\infty} \mu_{k+1}/\mu_k = 0$) while taking no more than a fixed number of Newton steps at each value of μ_k. In the case of LICQ, rapid convergence of this type has been investigated by Conn, Gould, and Toint [7], Benchakroun, Dussault, and Mansouri [4], Wright and Jarre [29], and Wright [26]. We anticipate that similar results will continue to hold when LICQ is replaced by MFCQ, because the central path continues to be smooth and the convergence domain (46) for Newton's method is similar in both cases. A detailed investigation of this claim and an analysis of the case in which strict complementarity fails to hold are left for future study.

References

1. I. Adler and R. D. C. Monteiro. Limiting behavior of the affine scaling continuous trajectories for linear programming problems. *Mathematical Programming*, 50:29–51, 1991.
2. M. Anitescu. On the rate of convergence of sequential quadratic programming with nondifferentiable exact penalty function in the presence of constraint degeneracy. Preprint ANL/MCS-P760-0699, Argonne National Laboratory, Argonne, Illinois, USA, 1999.
3. A.S. El Bakry, R.A. Tapia, T. Tsuchiya, and Y. Zhang. On the formulation and theory of newton interior point methods for nonlinear programming. *Journal of Optimization Theory and Applications*, 89(3):507–541, 1996.
4. A. Benchakroun, J.-P. Dussault, and A. Mansouri. A two parameter mixed interior-exterior penalty algorithm. *ZOR - Mathematical Methods of Operations Research*, 41:25–55, 1995.
5. R. H. Byrd, J. Ch. Gilbert, and J. Nocedal. A trust region method based on interior point techniques for nonlinear programming. Technical Report 2896, INRIA, Rocquencourt, France, 1996.
6. A. R. Conn, N. I. M. Gould, D. Orban, and Ph. L. Toint. A primal-dual trust-region algorithm for minimizing a non-convex function subject to bound and linear equality constraints. In preparation, 1999.

7. A. R. Conn, N. I. M. Gould, and Ph. L. Toint. A note on using alternative second-order models for the subproblems arising in barrier function methods for minimization. *Numerische Mathematik*, 68:17–33, 1994.
8. A. V. Fiacco and G. P. McCormick. *Nonlinear Programming: Sequential Unconstrained Minimization Techniques.* Wiley, New York, 1968. reprinted by SIAM Publications, 1990.
9. A. Forsgren and Ph. E. Gill. Primal-dual interior methods for nonconvex nonlinear programming. *SIAM Journal on Optimization*, 8(4):1132–1152, November 1998.
10. J. Gauvin. A necessary and sufficient regularity condition to have bounded multipliers in nonconvex programming. *Mathematical Programming*, 12:136–138, 1977.
11. David M. Gay, Michael L. Overton, and Margaret H. Wright. A primal-dual interior method for nonconvex nonlinear programming. Technical Report 97-4-08, Computing Sciences Research, Bell Laboratories, Murray Hill, NJ, July 1997.
12. N. I. M. Gould. On the accurate determination of search directions for simple differentiable penalty functions. *IMA Journal of Numerical Analysis*, 6:357–372, 1986.
13. A. J. Hoffman. On approximate solutions of systems of linear inequalities. *Journal of Research of the National Bureau of Standards*, 49:263–265, 1952.
14. F. A. Lootsma. Hessian matrices of penalty functions for solving constrained optimization. *Philips Res. Repts*, 24:322–331, 1969.
15. R. Mifflin. Convergence bounds for nonlinear programming algorithms. *Mathematical Programming*, 8:251–271, 1975.
16. W. Murray. Analytical expressions for the eigenvalues and eigenvectors of the Hessian matrices of barrier and penalty functions. *Journal of Optimization Theory and Applications*, 7:189–196, 1971.
17. Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming.* SIAM, Philadelphia, USA, 1994.
18. J. Nocedal and S. J. Wright. *Numerical Optimization.* Springer, New York, 1999.
19. J. M. Ortega and W. C. Rheinboldt. *Iterative solution of nonlinear equations in several variables.* Academic Press, New York and London, 1970.
20. D. Ralph and S. J. Wright. Superlinear convergence of an interior-point method despite dependent constraints. Preprint ANL.MCS-P622–1196, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, Ill., November 1996.
21. D. Ralph and S. J. Wright. Superlinear convergence of an interior-point method for monotone variational inequalities. In M. C. Ferris and J.-S. Pang, editors, *Complementarity and Variational Problems: State of the Art*, pages 345–385. SIAM Publications, Philadelphia, Penn., 1997.
22. M. H. Wright. Interior methods for constrained optimization. In *Acta Numerica 1992*, pages 341–407. Cambridge University Press, Cambridge, 1992.
23. M. H. Wright. Some properties of the Hessian of the logarithmic barrier function. *Mathematical Programming*, 67:265–295, 1994.
24. M. H. Wright. Ill-conditioning and computational error in interior methods for nonlinear programming. *SIAM Journal on Optimization*, 9:84–111, 1998.
25. S. J. Wright. Modifying SQP for degenerate problems. Preprint ANL/MCS-P699-1097, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, Ill., October 1997.
26. S. J. Wright. On the convergence of the Newton/log-barrier method. Preprint ANL/MCS-P681-0897, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, Ill., August 1997. Revised January, 1999.
27. S. J. Wright. Effects of finite-precision arithmetic on interior-point methods for nonlinear programming. Preprint ANL/MCS-P705-0198, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, Ill., January 1998.
28. S. J. Wright. Superlinear convergence of a stabilized SQP method to a degenerate solution. *Computational Optimization and Applications*, 11:253–275, 1998.
29. S. J. Wright and F. Jarre. The role of linear objective functions in barrier methods. *Mathematical Programming, Series A*, 84:357–373, 1998.